Quasi-Periodic, Periodic Waves, and Soliton Solutions for the Combined KdV-mKdV Equation

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By introducing the generalized Jacobi elliptic function, a new improved Jacobi elliptic function method is used to construct the exact travelling wave solutions of the nonlinear partial differential equations in a unified way. With the help of the improved Jacobi elliptic function method and symbolic computation, some new exact solutions of the combined Korteweg-de Vries-modified Korteweg-de Vries (KdV-mKdV) equation are obtained. Based on the derived solution, we investigate the evolution of doubly periodic and solitons in the background waves. Also, their structures are further discussed graphically.

Key words: Improved Jacobi Elliptic Function Method; Elliptic Equation; Generalized Jacobi Elliptic Functions; Combined KdV-mKdV Equation; Soliton Solutions.

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1. Introduction

Many phenomena in physics and engineering are described by nonlinear partial differential equations (NLPDEs). When we want to understand the physical mechanism of phenomena in nature, described by NLPDEs, exact solutions for the NLPDEs have to be explored. Thus methods for deriving exact solutions for the governing equations have to be developed. Studying exact solutions of NLPDEs has become one of the most important topics in mathematical physics. For instances, the nonlinear wave phenomena observed in fluid dynamics, plasma, and optical fiber are often modeled by the bell-shaped sech solutions and the kink-shaped tanh solutions. The availability of these exact solutions for those nonlinear equations can greatly facilitate the verification of numerical solutions. The stability analysis of the solutions of NLPDEs has a wide array of applications in many fields, which describe the motion of isolated waves, localized in a small part of space, such as in physics, in which applications extend over magneto fluid dynamics, water surface gravity waves, electromagnetic radiation reactions, and ion acoustic waves in plasmas. And also applications in biology, chemistry, and several other fields. These solutions may well describe various phenomena in physics and other fields, such as solitons and propagation with a finite speed, and thus they may give more insight into the physical aspects of the problems. The development of soliton theory clarifies the importance of seeking for exact solutions of NLPDEs. There has been noticeable progress in the study of soliton theory. In recent studies many powerful approaches were presented, such as the inverse scattering transform method [1], the Bäcklund transformation (BT) [2], the Darboux transformation (DT) [3, 4], the Hirota's bilinear homogeneous balance method [5, 6], the Jacobi elliptic function method [7 – 9], the tanh-function method [10 – 12], the variational iteration method [13], the sine-cosine method [14 – 16], the F-expansion method [17, 18], the Lucas Riccati method, the triangular Fibonacci method, the extended projective Riccati equation method [19], and so on. An important approach is the so-called mapping transformation method. The basic idea of the algorithm is that: for a given NLPDE

$$P(u, u_1, u_{x_i}, u_{x_i x_j}, \ldots) = 0, \tag{1}$$

where P is in general a polynomial function of its argument, and the subscripts denote the partial derivatives. By using the travelling wave transformation, (1) possesses the following ansatz,

$$u = u(\xi), \quad \xi = \sum_{i=0}^{m} k_i x_i, \tag{2}$$

where k_i, $i = 0,1,2,\ldots,m$ are all arbitrary constants. Substituting (2) into (1) yields an ordinary differential equation (ODE): $O(u(\xi),u(\xi)_\xi,u(\xi)_{\xi\xi},\ldots) = 0$. Then $u(\xi)$ is expanded into a polynomial in $g(\xi)$

$$u(\xi) = F(g(\xi)) = \sum_{i=0}^{n} a_i g^i(\xi), \tag{3}$$

where a_i are constants to be determined and n is fixed by balancing the linear term of the highest order with the nonlinear term in (1). If we suppose $g(\xi) = \tanh\xi$, $g(\xi) = \mathrm{sech}\xi$, and $g(\xi) = \mathrm{sn}\xi$ or $g(\xi) = \mathrm{cn}\xi$, respectively, then the corresponding approach is usually called the tanh-function method, the sech-function method, and the Jacobian-function method. Although the Jacobian elliptic function method is more improved than the tanh-function method and the sech-function method, the repeated calculations are often tedious since the different function $g(\xi)$ should be treated in a repeated way. The main idea of the mapping approach is that $g(\xi)$ is not assumed to be a specific function, such as tanh, sech, sn and cn, etc., but a solution of a mapping equation such as the Riccati equation ($g_\xi = g^2 + a_0$), or a solution of the cubic nonlinear Klein Gordon equation (${g^2}_\xi = a_4 g^4 + a_2 g^2 + a_0$), or a solution of the general elliptic equation ($g_\xi^2 = \sum_{i=0}^{4} a_i g^i$), where a_i, ($i = 0,1,\ldots,4$) are all arbitrary constants. Using the mapping relation (3) and the solutions of these mapping equations, one can obtain many explicit and exact travelling wave solutions of (1).

Nowadays, many exact solutions of NLPDEs can be written as a polynomial in several elementary or special functions which satisfy a first-order nonlinear ordinary differential equation (NLODE) with a sixth-degree nonlinear term. Recently, after Fan's unified algebraic method [20], which was based on a NLODE with a fourth-degree nonlinear term, Haung and Zhang [21] further extend Fan's unified algebraic method to a more general form, which possesses a sixth-degree nonlinear term, and derive more new exact solutions of NLPDEs. The aim of this paper, motivated by references [22–24], is to perform a first-order NLODE with sixth-degree nonlinear term which is, in nature, an extension of a type of elliptic equation into a new algebraic method to seek exact solutions for NLPDEs.

The rest of this paper is organized as follows: in the following section, we give the definition and properties of the generalized Jacobi elliptic functions. In sections 3 and 4, we describe the generalized improved Jacobi elliptic function method and apply this method to the combined KdV-mKdV equation. Finally, we conclude the paper and give some outlooks and comments.

2. Definition and Properties of Generalized Jacobi Elliptic Functions

In this section, we introduce the generalized Jacobi elliptic functions (GJEFs) and study some properties of these functions. We consider the hyper elliptic integral

$$y(x,k_1.k_2) = \int_0^x \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k_1^2t^2)(1-k_2^2t^2)}}. \tag{4}$$

We define the generalized Jacobi elliptic sine function as the inverse function $x = s(y,k_1,k_2)$, where y is an independent variable and k_1, k_2 $(0 \le k_2 \le k_1 \le 1)$ are two modulus of the generalized Jacobi elliptic functions. Similarly, $\sqrt{1-x^2}$, $\sqrt{1-k_1^2x^2}$, and $\sqrt{1-k_2^2x^2}$ are defined as the generalized Jacobi elliptic cosine function, the generalized Jacobi elliptic function of the third kind and the generalized Jacobi elliptic function of the fourth kind. They are expressed as

$$\begin{aligned} &\sqrt{1-x^2} = c(y,k_1,k_2),\\ &\sqrt{1-k_1^2x^2} = d_1(y,k_1,k_2),\\ &\sqrt{1-k_2^2x^2} = d_2(y,k_1,k_2). \end{aligned} \tag{5}$$

The generalized Jacobi elliptic functions possess the following properties of the triangular functions (we use the abbreviated notations $s(y) \equiv s(y,k_1,k_2)$, $c(y) \equiv s(y,k_1,k_2)$, …, etc.):

$$\begin{aligned} &c^2(y) = 1 - s^2(y), \quad d_1^2(y) = 1 - k_1^2 s^2(y),\\ &d_2^2(y) = 1 - k_2^2 s^2(y),\\ &k_1^2 d_2^2(y) - k_2^2 d_1^2(y) = k_1^2 - k_2^2,\\ &d_i^2(y) - k_i^2 c^2(y) = 1 - k_i^2, \quad (i = 1,2). \end{aligned} \tag{6}$$

The first derivatives of these functions are given by

$$\begin{aligned} &s'(y) = c(y)d_1(y)d_2(y),\\ &c'(y) = -s(y)d_1(y)d_2(y),\\ &d_1'(y) = -k_1^2 s(y)c(y)d_2(y),\\ &d_2'(y) = -k_2^2 s(y)c(y)d_1(y), \end{aligned} \tag{7}$$

Moreover, in the limiting case $k_2 \to 0$, the generalized Jacobi elliptic function reduced to the usual Jacobi elliptic functions

$$\begin{aligned} &s(y,k_1,0) \to \operatorname{sn}(y,k_1), \\ &c(y,k_1,0) \to \operatorname{cn}(y,k_1), \\ &d_1 c(y,k_1,0) \to \operatorname{dn}(y,k_1), \text{ and} \\ &d_2(y,k_1,0) \to 1. \end{aligned} \tag{8}$$

If $k_1 \to 1$, $k_2 \to 0$, we have

$$\begin{aligned} &s(y,1,0) \to \tanh(y), \\ &c(y,1,0) \text{ and } d_1(y,1,0) \to \operatorname{sech}(y), \text{ and} \\ &d_2(y,1,0) \to 1. \end{aligned} \tag{9}$$

Also, in the limiting case $k_1 \to 0$, $k_2 \to 0$, we have

$$\begin{aligned} &s(y,0,0) \to \sin(y), \quad c(y,0,0) \to \cos(y), \\ &d_1(y,0,0), \text{ and } d_2(y,0,0) \to 1. \end{aligned} \tag{10}$$

The generalized Jacobi elliptic functions can be expressed in terms of the standard Jacobi elliptic functions:

$$\begin{aligned} s(y,k_1,k_2) &= \frac{\operatorname{sn}(k_2' y,k)}{\sqrt{1-k_2^2+k_2^2\operatorname{sn}^2(k_2' y,k)}}, \\ c(y,k_1,k_2) &= \frac{k_2'\operatorname{cn}(k_2' y,k)}{\sqrt{1-k_2^2\operatorname{cn}^2(k_2' y,k)}}, \\ d_1(y,k_1,k_2) &= \frac{\sqrt{k_1^2-k_2^2}\operatorname{dn}(k_2' y,k)}{\sqrt{k_1^2-k_2^2\operatorname{dn}^2(k_2' y,k)}}, \\ d_2(y,k_1,k_2) &= \frac{\sqrt{k_1^2-k_2^2}}{\sqrt{k_1^2-k_2^2\operatorname{dn}^2(k_2' y,k)}}, \end{aligned} \tag{11}$$

with $k_2' = \sqrt{1-k_2^2}$, $k = \sqrt{(k_1^2-k_2^2)/(1-k_2^2)}$. From the double periodic properties of the Jacobi elliptic functions one can see that the generalized Jacobi elliptic functions are quasi-double periodic:

$$\begin{aligned} s\left(y+\frac{4\mathbf{K}(k)}{k_2'}\right) &= s\left(y+\frac{2\mathrm{i}\mathbf{K}(k')}{k_2'}\right) \\ &= \pm s(y), \\ c\left(y+\frac{4\mathbf{K}(k)}{k_2'}\right) &= c\left(y+\frac{2\mathbf{K}(k)+2\mathrm{i}\mathbf{K}(k')}{k_2'}\right) \\ &= \pm c(y), \\ d_1\left(y+\frac{2\mathbf{K}(k)}{k_2'}\right) &= d_1\left(y+\frac{4\mathrm{i}\mathbf{K}(k')}{k_2'}\right) \\ &= \pm d_1(y), \\ d_2\left(y+\frac{2\mathbf{K}(k)}{k_2'}\right) &= d_2\left(y+\frac{2\mathrm{i}\mathbf{K}(k')}{k_2'}\right) \\ &= \pm d_2(y), \end{aligned} \tag{12}$$

where $\mathbf{K}(k)$ is the complete elliptic integral of the first kind and $k' = \sqrt{1-k^2}$ [25–27].

3. Description of the Generalized Improved Jacobi Elliptic Function Method

The main idea of this method is to take full advantage of the elliptic equation that the generalized Jacobi elliptic functions satisfy. The desired elliptic equation read

$$\begin{aligned} &F'(\xi) = \sqrt{A_0 + A_2F^2(\xi) + a_4F^4(\xi) + A_6F^6(\xi)}, \\ &' \equiv \frac{\mathrm{d}}{\mathrm{d}\xi}, \end{aligned} \tag{13}$$

where $\xi \equiv \xi(x,t)$ and A_0, A_2, A_4, A_6 are constants.

Case 1: If $A_0 = 1$, $A_2 = -(1+k_1^2+k_2^2)$, $A_4 = k_1^2+k_2^2+k_1^2k_2^2$, and $A_6 = -k_1^2k_2^2$ then (13) has a solution $s(\xi,k_1,k_2)$.

Case 2: If $A_0 = 1-k_1^2-k_2^2+k_1^2k_2^2$, $A_2 = 2k_1^2+2k_2^2-3k_1^2k_2^2-1$, $A_4 = 3k_1^2k_2^2-k_1^2-k_2^2$, and $A_6 = -k_1^2k_2^2$ then (13) has a solution $c(\xi,k_1,k_2)$.

Case 3: If $A_0 = k_1^2-1-k_2^2+k_2^2k_1^{-2}$, $A_2 = 2k_2^2+2-k_1^2-3k_2^2k_1^{-2}$, $A_4 = 3k_2^2k_1^{-2}-k_2^2-1$, and $A_6 = -k_2^2k_1^{-2}$ then (13) has a solution $d_1(\xi,k_1,k_2)$.

Case 4: If $A_0 = k_2^2-1-k_1^2+k_1^2k_2^{-2}$, $A_2 = 2k_1^2+2-k_2^2-3k_1^2k_2^{-2}$, $A_4 = 3k_1^2k_2^{-2}-k_1^2-1$, and $A_6 = -k_1^2k_2^{-2}$ then (13) has a solution $d_2(\xi,k_1,k_2)$.

For a given NLPDE with u in two independent variables x, t, (1), we seek the travelling wave solution of (1) in the form $\xi = k(x+\omega t)$ where k is the wave number and ω the wave velocity. Then (1) is transformed to the ODE (3). We expand $u(x,t) = u(\xi)$ into a new polynomial of $F(\xi)$ in the form

$$u(\xi) = a_0 + \sum_{i=1}^{n}[a_iF^i(\xi) + b_iF^{i-1}(\xi)F'(\xi)]. \tag{14}$$

The processes take the following steps.

Step 1: Determine n in (14) by balancing the linear term(s) of the highest order with the nonlinear term(s) in (3).

Step 2: Substituting (14) with (13) into (3), then the left-hand side of (3) can be converted into a polynomial in $F(\xi)$. Setting each coefficient of the polynomial to zero yields a system of algebraic equations for a_0 a_1, $\dots$, a_n, b_1, $\dots$, b_n, k, and ω.

Step 3: Solving this system obtained in step 2, then a_0 a_1, $\dots$, a_n, b_1, $\dots$, b_n, k, and ω can be expressed by A_0, A_2, A_4, A_6. Substituting these into the generalized improved Jacobi elliptic function method (14), a general form of travelling wave solution of (1) can be obtained. In the following section, we apply this method to the combined KdV-mKdV equation to obtain new quasi-doubly periodic solution.

4. Application to the Combined KdV-mKdV Equation

We consider here the combined KdV-mKdV equation

$$u_t + 6\alpha u u_x + 6\beta u^2 u_x + \gamma u_{xxx} = 0, \qquad (15)$$

where α, β and γ are constants. This equation is widely used in various fields of physics such as solid-state physics, plasma physics, fluid physics and quantum field theory. Lou and Chen [28] found solitary wave solutions and sinusoidal wave solutions of (15) by using the mapping approach. Zhao et al. [29] got soliton-like solutions of (15) by applying the extended tanh method. Very recently Zhao et al. [30] and Sirendaoreji [31] obtained travelling wave solutions by means of new Riccati equation expansion method and a new auxiliary method.

According to step 1, we get $n = 2$ for u. In order to search for explicit solutions, we assume that (15) has the following formal solution:

$$u(\xi) = a_0 + a_1 F + a_2 F^2 + b_1 F' + b_2 F F', \qquad (16)$$

in which a_0, a_1, a_2, b_1, and b_2 are real constants to be determined. With the aid of Maple, substituting (16) along with (13) into (15) and setting each coefficient of $F^i F'^j$ ($j = 0,1;\ i = 1,2,\dots$) to zero, we get a set of over determined equations for a_0, a_1, a_2, b_1, k, and ω. Solving the system by the use of Maple, we obtain the following results:

Case 1:

$$\begin{aligned} &a_1 = b_1 = b_2 = 0,\\ &a_0 = \frac{-\alpha A_6 \pm k A_4\sqrt{-A_6\gamma\beta}}{2A_6\beta},\\ &a_2 = \pm 2k\sqrt{\frac{-A_6\gamma}{\beta}},\\ &\omega = \frac{k(3A_6\alpha^2 + k^2\beta\gamma(3A_4^2 - 8A_6A_6)}{2A_6\beta}. \end{aligned} \qquad (17)$$

Case 2:

$$\begin{aligned} &a_0 = -\frac{\alpha}{2\beta}, \quad a_1 = \pm k\sqrt{-\frac{A_4\gamma}{\beta}},\\ &a_2 = b_1 = b_2 = 0, \quad \omega = \frac{k(3\alpha^2 - 2k^2A_2\beta\gamma)}{2\beta}. \end{aligned} \qquad (18)$$

Equations (17) and (18) together with (16) are the solutions of the combined KdV-mKdV equation (15) if the appropriate parameters A_0, A_2, A_4, A_6 are given. We obtain the following solution of the combined KdV-mKdV equation (15):

$$u_1 = \frac{\alpha k_1k_2 \pm k(k_1^2 + k_2^2 + k_1^2k_2^2)\sqrt{\gamma\beta}}{-2k_1k_2\beta} \pm 2kk_1k_2\sqrt{\frac{\gamma}{\beta}}s^2(\xi, k_1, k_2), \qquad (19)$$

with $\xi = k\left[x + \frac{1}{2\beta}[-3\alpha^2 + k^2\beta\gamma(3(k_1^2k_2^{-2} + k_2^2k_1^{-2} + k_1^2k_2^2) - 2(1 + k_1^2 + k_2^2))]t\right]$,

$$u_2 = -\frac{\alpha}{2\beta} \pm k\sqrt{-\frac{(k_1^2 + k_2^2 + k_1^2k_2^2)\gamma}{\beta}}s(\xi, k_1, k_2), \qquad (20)$$

with $\xi = k\left[x + \frac{1}{2\beta}[3\alpha^2 + 2k^2\beta\gamma(1 + k_1^2 + k_2^2)]t\right]$,

$$u_3 = \frac{\alpha k_2k_1 \pm k(3k_1^2k_2^2 - k_1^2 - k_2^2)\sqrt{\gamma\beta}}{-2k_2k_1\beta} \pm 2kk_2k_1\sqrt{\frac{\gamma}{\beta}}c^2(\xi, k_1, k_2), \qquad (21)$$

with $\xi = k\left[x - \frac{1}{2\beta}[-3\alpha^2 + k^2\beta\gamma(3k_1^2k_2^{-2} + k_1^{-2}k_2^2 + k_1^2k_2^2) - 2(k_1^2 + k_2^2 + 3))]t\right]$,

$$u_4 = -\frac{\alpha}{2\beta} \pm k\sqrt{-\frac{(3k_1^2k_2^2 - k_1^2 - k_2^2)\gamma}{\beta}}c(\xi, k_1, k_2), \tag{22}$$

with $\xi = k\left[x + \frac{1}{2\beta}[3\alpha^2 - 2k^2\beta\gamma(2k_1^2 + 2k_2^2 - 3k_1^2k_2^2 - 1)]t\right]$,

$$u_5 = \frac{\alpha k_2^2 \pm k(k_1(1-2k_2^2) + 3k_2^2k_1^{-2})\sqrt{\gamma\beta}}{-2k_2^2\beta} \pm \frac{2kk_2}{k_1}\sqrt{\frac{\gamma}{\beta}}d_1^2(\xi, k_1, k_2), \tag{23}$$

with $\xi = k\left[x - \frac{1}{2\beta}[-3\alpha^2 + k^2k_1^2k_2^{-2}\beta\gamma(3(1 + k_2^4 + k_1^{-4}k_2^4) - 2k_2^2(1 + k_1^{-2} + k_2^2))]t\right]$,

$$u_6 = -\frac{\alpha}{2\beta} \pm k\sqrt{-\frac{(3k_2^2k_1^{-2} - k_2^2 - 1)\gamma}{\beta}}d_1(\xi, k_1, k_2), \tag{24}$$

with $\xi = k\left[x + \frac{1}{2\beta}[3\alpha^2 - 2k^2\beta\gamma(2k_2^2 + 2 - k_1^2 - 3k_2^2k_1^{-2})]t\right]$,

$$u_7 = \frac{\alpha k_1^2 \pm k(k_2(1-2k_1^2) + 3k_1^2k_2^{-2})\sqrt{\gamma\beta}}{-2k_1^2\beta} \pm \frac{2kk_1}{k_2}\sqrt{\frac{\gamma}{\beta}}d_2^2(\xi, k_1, k_2), \tag{25}$$

with $\xi = k\left[x - \frac{1}{2\beta}[-3\alpha^2 + k^2k_2^2k_1^{-2}\beta\gamma(3(1 + k_1^4 + k_2^{-4}k_1^4) - 2k_1^2(1 + k_2^{-2} + k_1^2))]t\right]$,

$$u_8 = -\frac{\alpha}{2\beta} \pm k\sqrt{-\frac{(3k_1^2k_2^{-2} - k_1^2 - 1)\gamma}{\beta}}d_2(\xi, k_1, k_2), \tag{26}$$

with $\xi = k[x + [3\alpha^2 - 2k^2\beta\gamma(2k_1^2 + 2 - k_2^2 - 3k_1^2k_2^{-2})]t/(2\beta)]$. The evolution graph of the quasi periodic solutions u_2, u_4, u_6, and u_8 with the positive sign are shown in Figures 1 – 4.

For $k_2 \to 0$, the above solutions degenerate to the well-known Jacobi elliptic wave solutions (doubly-periodic waves)

$$u_9 = -\frac{\alpha}{2\beta} \pm kk_1\sqrt{-\frac{\gamma}{\beta}}\mathrm{sn}\left(k\left[x + \frac{1}{2\beta}[3\alpha^2 + 2k^2(1+k_1^2)\beta\gamma]t\right], k_1\right), \tag{27}$$

$$u_{10} = -\frac{\alpha}{2\beta} \pm kk_1\sqrt{\frac{\gamma}{\beta}}\mathrm{cn}\left(k\left[x + \frac{1}{2\beta}[3\alpha^2 - 2k^2(k_1^2-1)\beta\gamma]t\right], k_1\right), \tag{28}$$

$$u_{11} = -\frac{\alpha}{2\beta} \pm k\sqrt{\frac{\gamma}{\beta}}\mathrm{dn}\left(k\left[x + \frac{1}{2\beta}[3\alpha^2 - 2k^2(2-k_1^2)\beta\gamma]t\right], k_1\right). \tag{29}$$

For $k_2 \to 0$, $k_1 \to 1$, the above solutions degenerate to the well-known solitary wave solutions (kink-shaped tanh and bell-shaped sech solutions)

$$u_{13} = -\frac{\alpha}{2\beta} \pm k\sqrt{-\frac{\gamma}{\beta}}\tanh\left(k\left[x + \frac{1}{2\beta}[3\alpha^2 + 4k^2\beta\gamma]t\right]\right), \tag{30}$$

$$u_{14} = -\frac{\alpha}{2\beta} \pm k\sqrt{-\frac{\gamma}{\beta}}\mathrm{sech}\left(k\left[x + \frac{1}{2\beta}[3\alpha^2 - 2k^2\beta\gamma]t\right]\right). \tag{31}$$

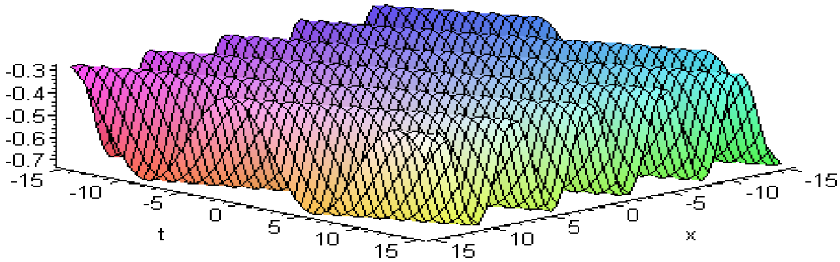

Fig. 1. Evolution graph of u_2 with the selection $\alpha = \beta = -\gamma = k = 1$, $k_1 = 0.2$, $k_2 = 0.1$.

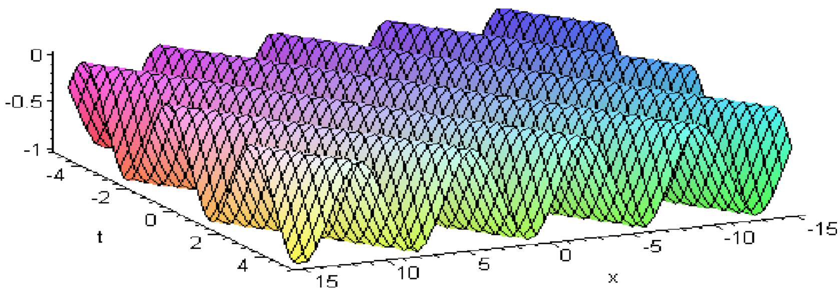

Fig. 2. Evolution graph of u_4 with the selection $\alpha = \beta = \gamma = k = 1$, $k_1 = 0.5$, $k_2 = 0.3$.

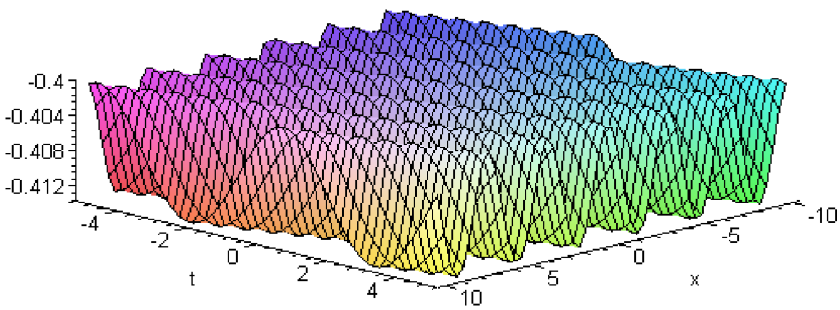

Fig. 3. Evolution graph of u_6 with the selection $\alpha = \beta = \gamma = k = 1$, $k_1 = 0.5$, $k_2 = 0.3$.

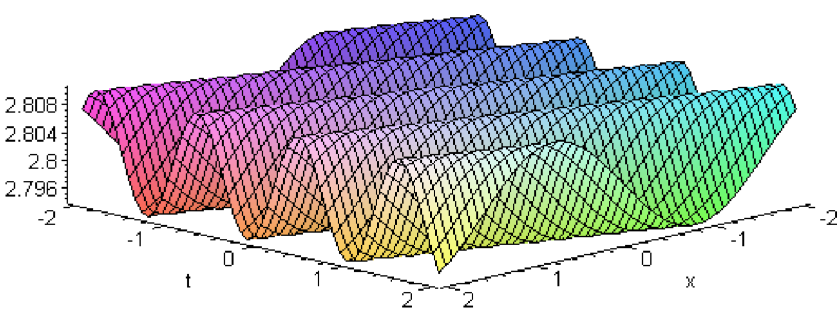

Fig. 4. Evolution graph of u_8 with the selection $\alpha = \beta = -\gamma = k = 1$, $k_1 = 0.2$, $k_2 = 0.1$.

5. Conclusion

There is no systematic way for solving (13). Nevertheless, this ansatz with four arbitrary parameters A_0, A_2, A_4, and A_6 is reasonable since its solution can be expressed in terms of functions, such as generalized Jacobi elliptic functions, that appear only in the nonlinear problems. In addition, these functions go back, in some limiting cases, to sn, cn, dn, tanh, and sech that describe the double periodic, solitary, and shock wave propagation. The values of the constants a_1 $(i = 0,1,2,\dots,n)$ and b_1 $(i = 1,2,\dots,n)$ in (14) depend crucially on the nature of differential equations whereas different types of their solutions can be classified in terms of A_0, A_2, A_4, and A_6 as shown in Cases 1 – 4. In this work, using generalized Jacobi elliptic functions and their limit cases, the quasi-periodic, periodic wave, and soliton solutions for the combined KdV-mKdV equation are obtained. Some of the properties of them are shown graphically. We believe one can apply this method to many other nonlinear differential equations in mathematical physics.

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